Mixed multiplier ideals and the irregularity of abelian coverings of the projective plane

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ABSTRACT

A formula for the irregularity of abelian coverings of the projective plane is established and some applications are presented.

1 INTRODUCTION

The initial intent of this study was to extend the formula for the cyclic multiple planes from [17] to the case where the branching curve C is not transverse to the line at infinity H_∞. In the transverse case, if S denotes a desingularization of the $\mathbb{Z}/n\mathbb{Z}$-cyclic covering of the plane associated to C and H_∞, then

$$q(S) = \sum_{\substack{\xi \text{ jumping number of } C \\ \xi \in 1/(n \wedge \deg C)\,\mathbb{Z},\ 0<\xi<1}} h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3+\xi \deg C) \otimes \mathcal{J}(\xi \cdot C)). \tag{1}$$

Hence, the irregularity is quasi-constant as a function of n, unlike what happens in the non transverse case, when, as we see in Example 4.4, the irregularity might be a degree 1 quasi-polynomial of n. To understand the difference and to extend the above formula to the non transverse case, we consider abelian instead of cyclic coverings. The role played by the multiplier ideals will be taken by the mixed multiplier ideals. Consequently, the goal of this paper is to apply the theory of mixed multiplier ideals to compute the irregularity of the abelian coverings of the projective plane.

If X is a smooth surface and $\mathfrak{a}_1, \dots, \mathfrak{a}_t \subset \mathcal{O}_X$ are non-zero ideal sheaves, the mixed multiplier ideal $\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t})$ varies with the rational vector $\boldsymbol{x} = (x^1, \dots, x^t) \in \mathbb{R}_+^t$. Proposition 2.2 and Proposition 2.7 assert that there is a set of hyperplanes called *jumping walls* with the following properties:

1. If the mixed multiplier ideal jumps, then the vector $\boldsymbol{x}$ crosses a jumping wall. Consequently, the fibres of the map $\boldsymbol{x} \mapsto \mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t})$ are finite unions of rational convex polytopes cut out by the jumping walls.
2. The jumping walls are determined by the *jumping numbers* of the *simple complete relevant ideals* (see Definition 2.6) associated to the ideals $\mathfrak{a}_i$.

These results together with O. Zariski's original idea introduced in [23] enable us to generalize formula (1) to abelian coverings of the projective plane. Such a covering induces a partition of the branching curve, and the irregularity is expressed in Theorem 3.9 as a linear combination of superabundances of linear systems defined in terms of some mixed multiplier ideals associated to this partition. There exists a natural map from the Galois group characters of the covering to the first orthant appearing in the definition of the mixed multiplier ideals. The coefficient of each superabundance represents the number of characters that lie in the intersection of the jumping walls associated to the corresponding mixed multiplier ideals. We refer the reader to Theorem 3.9 for the precise formula and note here that it could be easily extended along the lines of Vaquié's paper [22], to coverings of smooth surfaces.

The proof of Theorem 3.9 occupies §3. In §4, the last part of the paper, some applications are presented including E. Hironaka's result from [6] concerning the asymptotic behaviour of the irregularity of the abelian coverings, the discussion of the general cyclic coverings, and the computation of the irregularity of the Hirzebruch surfaces constructed in [7]—abelian coverings of the plane branched along configurations of lines, *i.e.* line arrangements. F. Hirzebruch mainly deals with three arrangements and obtains three families of surfaces with the covering group, for each family, a certain power of $\mathbb{Z}/n\mathbb{Z}$. For the three most interesting examples, one in each family, namely those with $c_1^2 = 3c_2$, the computation of the irregularity was performed by N.-M. Ishida in [9]. In [14], A. Libgober computed the irregularity for two of the three families for general n. We find again one of Libgober's results, slightly correct the second, see Proposition 4.8, and perform the computation for the third family.

In [1], N. Budur has obtained a general formula for the Hodge numbers $h^{0,q}$, $0 \leq q \leq n$, of the abelian coverings of a smooth variety of dimension n. His proof is based on the theory of local systems of rank 1, and the formula is expressed in terms of the number of certain rational points inside convex polytopes (see [1, Theorem 1.3, Theorem 1.7]). A. Libgober previously established in [14, § 3.1] a formula for the irregularity of abelian coverings of the plane, his technique being based on mixed Hodge structures. The computations, mentioned above for the families of Hirzebruch surfaces, used this formula. His formula and ours bear clear resemblances; it is a sum of superabundances of linear systems expressed in terms of quasiadjunction ideals (see [15] for the relation between the quasiadjunction ideals and the multiplier ideals) with coefficients given by quasiadjunction polytopes.

2 Mixed multiplier ideals and jumping walls

In this section we define and characterize the jumping walls associated to mixed multiplier ideals. We start by briefly recalling the notions of multiplier ideals and mixed multiplier ideals for ideal sheaves on a smooth surface following [10].

Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X and let $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$ with $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$. If ξ is a positive rational number, then the multiplier ideal associated to ξ and $\mathfrak{a}$ is defined as

$$\mathcal{J}(\mathfrak{a}^{\xi}) = \mu_* \mathcal{O}_Y(K_\mu - \lfloor \xi F \rfloor).$$

Now, for the analogous notion for several ideals, let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subset \mathcal{O}_X$ be non-zero ideals and $\mu : Y \to X$ a common log resolution of the ideals $\mathfrak{a}_i$ with $\mathfrak{a}_i \cdot \mathcal{O}_X = \mathcal{O}_Y(-F_i)$ and

$\sum_i F_i + \mathrm{except}(\mu)$ having simple normal crossing support. If $\xi_1, \ldots, \xi_t$ are positive rational numbers, then the *mixed multiplier ideal* associated to the ξ_i and the $\mathfrak{a}_i$ is

$$\mathcal{J}(\mathfrak{a}_1^{\xi_1} \cdots \mathfrak{a}_t^{\xi_t}) = \mu_* \mathcal{O}_Y(K_\mu - \lfloor \xi_1 F_1 + \cdots + \xi_t F_t \rfloor).$$

DEFINITION-LEMMA (SEE [10], LEMMA 9.3.21). *Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X and let $P \in X$ be a fixed point in the support of $\mathfrak{a}$. Then there is an increasing sequence of positive rational numbers $\xi_j = \xi_j(\mathfrak{a}, P)$ such that for every $\xi \in [\xi_j, \xi_{j+1})$,*

$$\mathcal{J}(\mathfrak{a}^{\xi_j}) = \mathcal{J}(\mathfrak{a}^{\xi}) \supset \mathcal{J}(\mathfrak{a}^{\xi_{j+1}}).$$

The rational numbers ξ_j are called the jumping numbers of the ideal sheaf $\mathfrak{a}$ at P.

The multiplier ideals and the jumping numbers are defined similarly in the context of effective $\mathbb{Q}$-divisors. By [10, Proposition 9.2.28], if C is a general element of the ideal sheaf $\mathfrak{a}$ and ξ is a positive rational less than 1, then $\mathcal{J}(\xi \cdot C) = \mathcal{J}(\mathfrak{a}^{\xi})$. Moreover, for any integer divisor C through a point P, the jumping numbers of C at P are periodic and determined by the ones lying in the unit interval $[0, 1)$. Similarly, the jumping numbers of an ideal sheaf $\mathfrak{a}$ at P are periodic and determined by the ones lying in the interval $[0, 2]$. We refer the reader to [10, Example 9.3.24] for more ample details.

For the remainder of this section we consider $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subset \mathcal{O}_X$ non-zero ideals such that the subscheme defined by each $\mathfrak{a}_i$ is zero dimensional and supported at a fixed point $P \in X$. We want to study the behaviour of the mixed multiplier ideal $\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t})$ as $\boldsymbol{x} = (x^1, \ldots, x_t)$ varies in the first orthant. If $\mu : Y \to X$ is a log resolution defined as before, we shall denote by E_α the strict transforms of the exceptional divisors seen on Y. There exists effective divisors B_α on Y such that (B_α) is the dual basis to $(-E_\alpha)$ of the lattice $\Lambda_\mu = \bigoplus_\alpha \mathbb{Z} E_\alpha$ with respect to the intersection form on Y. The basis (B_α) is called the *branch basis* of the resolution.

Next we want to define the notion of relevant divisors. We follow [20] but see also [4].

Definition 2.1. Let $\mathfrak{a} \subset \mathfrak{m}_P$. A strict transform E_ρ in a log resolution of $\mathfrak{a}$ is called a *relevant divisor* of $\mathfrak{a}$ at P if either

$$E_\rho \cdot (E_\rho^0) \geq 3, \tag{2}$$

where $E_\rho^0 = (\mu^* C)_{red} - E_\rho$ with C the curve defined by a general element of $\mathfrak{a}$, or E_ρ corresponds to an arrowhead vertices of the augmented Enriques tree of C at P. The index ρ will be referred to as a *relevant position.*

Note that the difference with respect to the notion introduced in [20] comes from the fact that we consider jumping numbers associated to ideal sheaves. For example, for the ideal of a knot, the exceptional divisor becomes a relevant divisor.

The set of relevant positions of $\mathfrak{a}$ at P will be denoted by $\mathfrak{R} = \mathfrak{R}_P(\mathfrak{a})$. The following proposition stresses the importance of the relevant divisors, or positions, in the computation of mixed multiplier ideals. It will further lead us to the notion of jumping walls associated to the ideal sheaf $\mathfrak{a}_1 \cdots \mathfrak{a}_t$ at P.

Proposition 2.2. *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_t \subset \mathcal{O}_X$ be non-zero ideals such that the subscheme defined by each $\mathfrak{a}_i$ is zero dimensional and supported at a fixed point $P \in X$. Let $\mu : Y \to X$ be a log resolution of $\mathfrak{a}$ and $\mathfrak{R}$ the set of relevant positions of $\mathfrak{a}$ at P. It x^i are positive rational numbers, then*

$$\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t}) = \mu_* \mathcal{O}_Y \left(K_\mu - \sum_{\rho \in \mathfrak{R}} \left\lfloor \sum_i x^i e_i^\rho \right\rfloor E_\rho \right),$$

where for every i, $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum_\alpha e_i^\alpha E_\alpha)$.

Proof. Consider $\boldsymbol{y} = c\boldsymbol{x}$ with $c \in [0,1]$. If $c = 1$ then $\boldsymbol{y} = \boldsymbol{x}$ and as c decreases, the coefficients $\lfloor \sum_i y^i e_i^\alpha \rfloor$ decrease by discrete jumps behind. More precisely, there is a finite sequence of rationals $0 = c_{g+1} < c_g < c_{g-1} < \cdots < c_1 < c_0 = 1$ with the following properties holding for any $0 \leq l \leq g$:

1) for any $c \in [c_{l+1}, c_l)$, and for any $\alpha \notin \mathfrak{R}$, $\lfloor c_{l+1} \sum_i x^i e_i^\alpha \rfloor = \lfloor c \sum_i x^i e_i^\alpha \rfloor$;

2) there exists $\mathfrak{B}(l)$ disjoint from $\mathfrak{R}$ such that for any $\beta \in \mathfrak{B}(l)$,

$$\left\lfloor c_{l+1} \sum_i x^i e_i^\beta \right\rfloor = \left\lfloor c_l \sum_i x^i e_i^\beta \right\rfloor - 1 = c_l \sum_i x^i e_i^\beta - 1;$$

3) for any $\alpha \notin \mathfrak{B}(l) \cup \mathfrak{R}$, $\lfloor c_{l+1} \sum_i x^i e_i^\alpha \rfloor = \lfloor c_l \sum_i x^i e_i^\alpha \rfloor$.

Set

$$\Delta_l = -\sum_{\alpha \notin \mathfrak{R}} \left\lfloor c_l \sum_i x^i e_i^\alpha \right\rfloor E_\alpha - \sum_{\rho \in \mathfrak{R}} \left\lfloor \sum_i x^i e_i^\rho \right\rfloor E_\rho.$$

To end the proof, it is sufficient to show that $\mu_* \mathcal{O}_Y(K_\mu + \Delta_{l+1}) = \mu_* \mathcal{O}_Y(K_\mu + \Delta_l)$ for any $0 \leq l < g$. Set $\Gamma = \sum_{\beta \in \mathfrak{B}(l)} E_\beta$. We have the following:

Claim. For any $\Gamma' \subset \Gamma$ and $E_\gamma \subset \Gamma'$ an irreducible component,

$$\mu_* \mathcal{O}_Y(K_\mu + \Delta_i + \Gamma' - E_\gamma) = \mu_* \mathcal{O}_Y(K_\mu + \Delta_i + \Gamma').$$

We justify the claim only when x^i are less than 1. The general case is similar, but one needs to consider the general form of [10, Proposition 9.2.28]. Let $C_1, \dots, C_t$ be the curves defined by general elements in $\mathfrak{a}_i$. Using *1)* and *2)* above we have

$$\begin{aligned}
-\Delta_l \cdot E_\gamma &\geq \sum_\alpha \left\lfloor c_l \sum_i x^i e_i^\alpha \right\rfloor E_\alpha \cdot E_\gamma \\
&> \sum_{\beta \in \mathfrak{B}(l)} c_l \sum_i x^i e_i^\beta E_\beta \cdot E_\gamma + \sum_{\alpha \notin \mathfrak{B}(l)} \Big(c_l \sum_i x^i e_i^\alpha - 1 \Big) E_\alpha \cdot E_\gamma \\
&\quad + \sum_i \Big(c_l x^i - 1 \Big) \widetilde{C}_i \cdot E_\gamma \\
&= c_l \sum_i x^i \mu^* C_i \cdot E_\gamma - ((\mu^* C)_{red} - \Gamma) \cdot E_\gamma.
\end{aligned}$$

Hence

$$(\Delta_l + \Gamma' - E_\gamma) \cdot E_\gamma < ((\mu^* C)_{red} - \Gamma) \cdot E_\gamma + (\Gamma' - E_\gamma) \cdot E_\gamma \leq E_0^\gamma \cdot E_\gamma \leq 2 \tag{3}$$

since $\gamma \notin \mathfrak{R}_P$. Now, tensoring the structure sequence of E_γ in Y with $\mathcal{O}_Y(K_\mu + \Delta_l + \Gamma')$ and pushing it down to X, we get the exact sequence

$$0 \to \mu_* \mathcal{O}_Y(K_\mu + \Delta_l + \Gamma' - E_\gamma) \to \mu_* \mathcal{O}_Y(K_\mu + \Delta_l + \Gamma') \to H^0(E_\gamma, K_{E_\gamma} + (\Delta_l + \Gamma' - E_\gamma)|_{E_\gamma}).$$

The last term vanishes by (3) justifying the claim.

From the properties *2)* and *3)*, $\Delta_{l+1} = \Delta_l + \Gamma$. By repeatedly using the claim, we obtain the result. □

Next, we want to define the *jumping walls* associated to the mixed multiplier ideals $\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t})$ when $\boldsymbol{x} = (x^1, \ldots, x^t)$ varies in the first orthant. The idea is that by the previous result, such a mixed multiplier ideal varies only when the point $\boldsymbol{x}$ crosses certain hyperplanes defined by equations corresponding to relevant positions. The defining equation of such a hyperplane is of the form

$$\sum_{i=1}^{t} x^i e_i^\rho = r, \tag{4}$$

with $\rho \in \mathfrak{R}$ and r a positive integer.

Definitions 2.3. A *relevant value* associated to the relevant position $\rho \in \mathfrak{R}$ of the ideal $\mathfrak{a}_1 \cdots \mathfrak{a}_t$ is a positive integer r such that there may be found a point $\boldsymbol{y}$ in the hyperplane $H : \sum_{i=1}^t x^i e_i^\rho = r$ and a neighbourhood V of $\boldsymbol{y}$ with the property that the mixed multiplier ideal $\mathcal{J}(\mathfrak{a}_1^{x^1} \cdots \mathfrak{a}_t^{x^t})$ corresponding to $\boldsymbol{x} \in V$, changes if and only if $\boldsymbol{x}$ crosses H. The pair (ρ, r) is called a *relevant pair* and the hyperplane H a *jumping wall.*

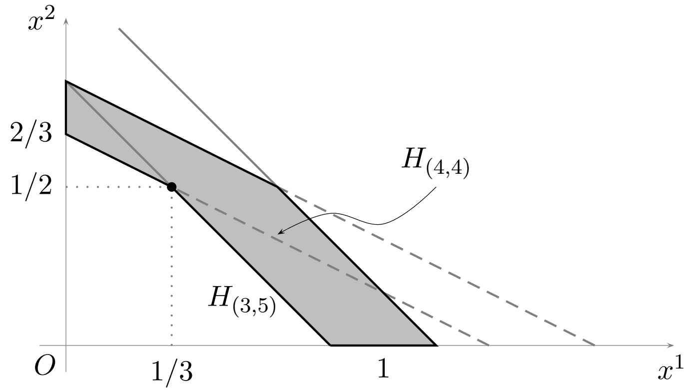
When we speak of a relevant value, we mean a positive integer which is the relevant value associated to a certain relevant position. Of course, such a value might be associated to many relevant positions, but the position we refer to will be clearly identified in the context.

Remark 2.4. If $\mathfrak{a}$ is a simple complete ideal, the relevant values associated to the relevant position ρ are the integers ξe_ρ, where e_ρ is the coefficient of the strict transform E_ρ in the minimal log resolution of $\mathfrak{a}$ and ξ runs over all the jumping numbers contributed by ρ. We refer the reader to [11, 18] for a formula producing all these jumping numbers.

Example 2.5. Let $\mathfrak{a}_1 = (u^3, v^2)$ and $\mathfrak{a}_2 = (u^6, v^2)$ be ideals in $\mathbb{C}[u, v]$. Let E_1, E_2 and E_3 be the exceptional divisors necessary for the minimal log resolution of $\mathfrak{a}_1$ and let E_4 be the supplementary exceptional divisor necessary for finishing the minimal log resolution of $\mathfrak{a}_2$. Clearly, if C_i are general elements in each $\mathfrak{a}_i$, then $\mu^*(C_1 + C_2) = \widetilde{C}_1 + \widetilde{C}_2 + 4E_1 + 7E_2 + 12E_3 + 9E_4$. The divisors E_3 and E_4 are the only relevant divisors. Then 5 and 7 are the first relevant values associated to the relevant divisor E_3 with the jumping walls $H_{(3,r)} : 6x^1 + 6x^2 = r$, $r = 5, 7$. Moreover, 4 and 5 are the first relevant values associated to E_4 with the jumping walls $H_{(4,r)} : 3x^1 + 6x^2 = r$, $r = 4, 5$.

The point $\boldsymbol{y}$ from the definition of the jumping wall $H_{(4,4)}$ can by any point on $H_{(4,4)} \cap \mathbb{R}_+^2$ with $y^1 < 1/3$. The other points in the intersection do not satisfy the property in the definition of the relevant value. If $y^1 > 1/3$ then on a sufficiently small neighbourhood of $\boldsymbol{y}$, the mixed multiplier ideal $\mathcal{J}(x^1 C_1 + x^2 C_2)$ equals the maximal ideal (u, v). If $\boldsymbol{y} = (1/3, 2/3)$

then the multiplier ideal also changes when it crosses the wall $H_{(3,5)} : 6x^1 + 6x^2 = 5$. In the figure above, if $\boldsymbol{x}$ lies in the open shaded polygon, then the mixed multiplier ideal equals the maximal ideal.



For practical reasons, what we have to do next is to determine a relatively small set of candidates for the relevant values associated to ρ.

Definition 2.6. Let ρ be a relevant position for the ideal $\mathfrak{a}$ and μ a log resolution. The relevant ideal associated to $\mathfrak{a}$ and ρ is the simple complete ideal $\mu_*\mathcal{O}_Y(-B_\rho)$, where B_ρ is the *rho* element in the branch basis of the resolution.

Proposition 2.7. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_t \subset \mathcal{O}_X$ be non-zero ideals such that the subscheme defined by each $\mathfrak{a}_i$ is zero dimensional and supported at a fixed point $P \in X$. Let $\mu : Y \to X$ be a log resolution of $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_t$ with (B_α) the branch basis of the resolution. Then the set of relevant values associated to the relevant position ρ is contained in the set of relevant values associated to ρ of the relevant ideal $\mu_*\mathcal{O}_Y(-B_\rho)$.*

Proof. It is sufficient to consider the case $t \geq 2$. Let ρ_0 be a relevant position and r a relevant value with $H : \sum_{i=1}^t x^i e_i^{\rho_0} = r$ the corresponding hyperplane. The point $\boldsymbol{y}$ may be chosen such that H is the only jumping hyperplane containing it. It is here that we need $t \geq 2$. Using Proposition 2.2, since

$$\mu_*\mathcal{O}_Y(K_\mu - \left\lfloor \sum_i y^i F_i \right\rfloor) \subset \mu_*\mathcal{O}_Y(K_\mu - \left\lfloor \sum_i y^i F_i \right\rfloor + E_{\rho_0}) = \mu_*\mathcal{O}_Y(K_\mu - \left\lfloor \sum_i (1-\varepsilon) y^i F_i \right\rfloor)$$

with $0 < \varepsilon \boldsymbol{l} 1$, we get that

$$\mu_*\mathcal{O}_Y(K_\mu - \sum_{\rho \in \mathfrak{R}} r^\rho E_\rho) \subset \mu_*\mathcal{O}_Y(K_\mu - \sum_{\rho \neq \rho_0} r^\rho E_\rho - (r-1)E_{\rho_0}).$$

Setting $K_\mu = \sum_\alpha k^\alpha E_\alpha$ and $\mathfrak{b} = \mu_*\mathcal{O}_Y(-\sum_{\rho \neq \rho_0} (r^\rho - k^\rho) E_\rho)$, it follows that

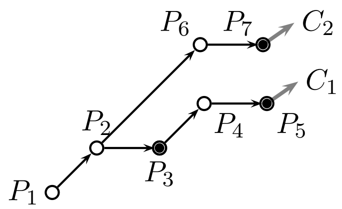
$$\mathfrak{b} \cap \mu_*\mathcal{O}_Y((k^{\rho_0} - r)E_{\rho_0}) \subset \mathfrak{b} \cap \mu_*\mathcal{O}_Y((k^{\rho_0} - r + 1)E_{\rho_0}),$$

i.e. that $\mu_*\mathcal{O}_Y((k^{\rho_0} - r)E_{\rho_0}) \subset \mu_*\mathcal{O}_Y((k^{\rho_0} - r + 1)E_{\rho_0})$. Now, set $\mathfrak{q} = \mu_*\mathcal{O}_Y(-B_{\rho_0})$. The Enriques tree associated to $\mu' : Y' \to X$, the minimal log resolution of $\mathfrak{q}$, is the path from the root to the vertex P_{ρ_0} of the Enriques tree associated to $\mathfrak{a}$. Let $\mathfrak{V}'$ be the set of vertices of this path and $\mathfrak{R}' \subset \mathfrak{R} \cap \mathfrak{V}'$ the set of relevant positions. If $\mathfrak{q} \cdot \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-\sum_{\alpha \in \mathfrak{V}'} e^{\alpha} E_{\alpha})$, let $\mathfrak{R}'' \subset \mathfrak{R}'$ be the subset of relevant positions such that for any $\rho \in \mathfrak{R}''$, re^{ρ}/e^{ρ_0} is an integer. Then, using again Proposition 2.2 and the previous strict inclusion,

$$\begin{aligned}\mathcal{J}(\mathfrak{q}^{r/e^{\rho_0}}) &= \mu_* \mathcal{O}_{Y'}\left(K_{\mu'} - \sum_{\rho \in \mathfrak{R}'} \frac{re^{\rho}}{e^{\rho_0}} E_{\rho}\right) = \mu_* \mathcal{O}_{Y'}\left(\sum_{\rho \in \mathfrak{R}'} \left(k^{\rho} - \frac{re^{\rho}}{e^{\rho_0}}\right) E_{\rho}\right) \\ &\subset \mu_* \mathcal{O}_{Y'}\left(\sum_{\rho \in \mathfrak{R}' \smallsetminus \mathfrak{R}''} \left(k^{\rho} - \frac{re^{\rho}}{e^{\rho_0}}\right) E_{\rho} + \sum_{\rho \in \mathfrak{R}''} \left(k^{\rho} - \frac{re^{\rho}}{e^{\rho_0}} + 1\right) E_{\rho}\right) = \mathcal{J}(\mathfrak{q}^{(1-\varepsilon)r/e^{\rho_0}}),\end{aligned}$$

where $0 < \varepsilon \boldsymbol{l} 1$. Hence r/e^{ρ_0} is a jumping number of $\mu_* \mathcal{O}_Y(-B_{\rho_0})$ associated to the relevant position ρ_0. $\square$

Example 2.8. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be simple complete ideals supported at P. We want to show that for a relevant position ρ, the set of relevant values of the ideal $\mu_* \mathcal{O}_Y(-B_{\rho})$ are indeed needed, *i.e.* that the union of the sets of of relevant values of each ideal $\mathfrak{a}_i$ associated to ρ is not sufficient. Let C_1 and C_2 be two unibranch curves, general elements in $\mathfrak{a}_1$ and $\mathfrak{a}_2$, and let the associated augmented Enriques tree of the minimal log resolution of $C_1 + C_2$ be as in the figure below.



We have $\mu^*(C_1 + C_2) = \widetilde{C}_1 + \widetilde{C}_2 + 6E_1 + 10E_2 + 18E_3 + 19E_4 + 38E_5 + 11E_6 + 22E_7$. The relevant positions are indicated by the black vertices: 3, 5 and 7. The jumping numbers of $\mathfrak{a}_1$ contributed by E_3 are $(5 + 6k)/12$, with $k \in \mathbb{N}$. But the three first jumping numbers of $\mathfrak{a}_1 \mathfrak{a}_2$ are $5/18$, $7/18$ and $8/18$. Hence the relevant values 7 and 8 are not among the relevant values of $\mathfrak{a}_1$ associated to the relevant position 3. Of course, the well known jumping numbers of the ideal $\mu_* \mathcal{O}_Y(-B_3)$ are $(2a + 3b)/6$, with a and b positive integers.

3 The irregularity of the abelian covering of the projective plane

In this section we state and prove in Theorem 3.9 a formula for the irregularity of the standard abelian coverings of the plane. To be able to express it, we start by summarizing the definition and some properties of these coverings in a form convenient for further use. Then, using the jumping walls in the context of plane curves, we introduce the notion of distinguished faces, leading notion in formula (7).

3.1 Abelian coverings

Let $\pi : Y \to X = Y/G$ be a Galois covering with abelian Galois group G. It is well known that $\pi_*\mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_X$-algebras and that $Y \simeq \mathbf{Spec}_{\mathcal{O}_X}(\pi_*\mathcal{O}_Y)$. In addition, if Y is normal and X is smooth, π is flat and consequently $\pi_*\mathcal{O}_Y$ is locally free of rank n. The action of G on $\pi_*\mathcal{O}_Y$ decomposes it into the direct sum of eigen line bundles associated to the characters $\chi \in \widehat{G} = \mathrm{Hom}(G, \mathbb{S}^1)$,

$$\pi_*\mathcal{O}_Y = \mathcal{O}_X \oplus \bigoplus_{\chi\in\widehat{G},\chi\neq 1} \mathcal{L}_\chi^{-1}.$$

The action of G on $\mathcal{L}_\chi^{-1}$ is the multiplication by χ.

Let $\chi_1, \ldots, \chi_s \in \widehat{G}$ such that the group of characters is the direct sum of the cyclic subgroups generated by $\chi_1, \ldots, \chi_s$. Let $n_1, \ldots, n_s$ be their orders. In [19, Proposition 2.1] it is shown that the ring structure of $\pi_*\mathcal{O}_Y$, and hence Y, are determined by the following linear equivalences or isomorphisms. For every $1 \le j \le s$,

$$n_j L_{\chi_j} \sim \sum_{f\in\mathfrak{F}} \frac{n_j f(\chi_j)^\bullet}{m_f} B_f, \tag{5}$$

where: 1) the set $\mathfrak{F}$ consists of all group epimorphisms from $\widehat{G}$ to different $\mathbb{Z}/m\mathbb{Z}$; 2) the curve $B_f \subset X$ with $f \in \mathfrak{F}$ is the sub-divisor of the branch locus defined set-theoretically as $\pi(R_f)$, with R_f the union of all the components D of the ramification locus associated to the group epimorphism f; 3) the integer $a^\bullet$ denotes the smallest non-negative integer in the equivalence class $a \in \mathbb{Z}/m$ (each time the integer m being understood from the context).

Remark. If D is a component of the ramification locus, since Y is normal and X smooth, D is 1-codimensional. The *inertia subgroup* $H \subset G$ and a character $\psi \in \widehat{H}$—the induced representation of H on the cotangent space to Y at D—that generates $\widehat{H}$ are associated to D. Dualizing the inclusion $H \subset G$, such a pair (H, ψ) is equivalent to a group epimorphism $f\colon \widehat{G} \to \mathbb{Z}/m_f$, where $m_f = |H|$.

Following [19], the line bundles $\mathcal{L}_{\chi_j}$, $1 \le j \le s$, and the divisors B_f, $f \in \mathfrak{F}$, are called a set of *reduced building data* for the covering. In case X is compact, the covering is uniquely determined by the isomorphisms (5), up to isomorphisms of abelian coverings. It is to be noticed that if $\chi = \chi_1^{a_1} \cdots \chi_s^{a_s} \in \widehat{G}$, then

$$L_\chi \sim \sum_{j=1}^{s} a_j L_{\chi_j} - \sum_{f\in\mathfrak{F}} \left\lfloor \sum_{j=1}^{s} \frac{a_j f(\chi_j)^\bullet}{m_f} \right\rfloor B_f. \tag{6}$$

So, to a normal covering $\pi : Y \to X$ of a smooth variety X, a set of reduced building data is associated satisfying the relations (5). Conversely, starting with a set of reduced building data and relations (5) an abelian covering is constructed which will be called a *standard abelian covering.*

Notation 3.1. In the sequel, we fix the notation $S(\boldsymbol{n}, M, \boldsymbol{C}, H_\infty)$ for the abelian covering of the plane constructed as follows. Let $C \subset \mathbb{P}^2$ be a reduced curve and $H_\infty \subset \mathbb{P}^2$ a line called the line at infinity. The set of reduced building data consists of

- the line bundles $\mathcal{L}_{\chi_j} = \mathcal{O}_{\mathbb{P}^2}(\left\lceil \sum_{i=1}^t \mu_j^i d_i/n_j \right\rceil)$, $1 \le j \le s$,
- the line H_∞ and the curves $C_i \subset C$ of degree d_i, $1 \le i \le t$, such that $C = \sum_i C_i$,
- the linear equivalences $n_j L_{\chi_j} \sim \sum_{i=1}^t \mu_j^i C_i + (\lceil \mu_j d_j/n_j \rceil n_j - \mu_j d_j) H_\infty$, $1 \le j \le s$.

The covering $S \to \mathbb{P}^2$ thus obtained has Galois group $\oplus_{j=1}^s \mathbb{Z}/n_j\mathbb{Z}$ and depends on the line at infinity and the $r \times s$ matrix $M = [\mu_j^i]$ with non negative integer entries. $\boldsymbol{C}$ is the list of curves $(C_1, \ldots, C_t)$ and $\boldsymbol{n}$ the s-vector $(n_1, \ldots, n_s)$ defining the covering group. The list of curves $\boldsymbol{C}$ such that $C = \sum_i C_i$ will be referred to as a *partition* of C.

Remark 3.2. In [6] the construction of the abelian coverings Σ_n that are studied is different from the one presented above. The construction is based on the composition of the Hurewich epimorphism with the morphism given by the change of constants in the homology groups

$$\pi_1(U) \longrightarrow H_1(U, \mathbb{Z}) \longrightarrow H_1(U, \mathbb{Z}/n\mathbb{Z}).$$

Here $U = \mathbb{P}^2 \setminus (C \cup H_\infty) = \mathbb{C}^2 \setminus C$, with $C = \sum_{j=1}^s C_j$ the decomposition of C into *irreducible components*. It is known that there exists an exact sequence (see [2, Proposition 1.3])

$$\mathbb{Z} \xrightarrow{\iota} \bigoplus_{j=0}^{s} \mathbb{Z} \longrightarrow H_1(U, \mathbb{Z}) \longrightarrow 0$$

with $\iota(1) = g_0 + \sum_{j=1}^s d_j g_j$, $g_0 = (1, 0, \ldots)$ and so on. It follows that the epimorphism corresponds to a Galois unbranched covering $V \to U$ with group $H_1(U, \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^s$. By the existence theorem of Grauert and Remmert [5], this covering extends to a unique normal abelian covering $\pi : \Sigma_n \to \mathbb{P}^2$, *i.e.* such that $\pi^{-1}(U) = V$. It turns out that Σ_n is the normalization of the standard covering $S((n, \ldots, n), I_s, \boldsymbol{C}, H_\infty)$ introduced in Notation 3.1, *i.e.* for which the linear equivalences (5) are given by $nL_{\chi_j} \sim C_j + (\lceil d_j/n \rceil n - d_j) H_\infty$.

3.2 Jumping walls and distinguished faces of a curve endowed with a partition

Let C be a reduced plane curve endowed with a partition $\boldsymbol{C} = (C_1, \ldots, C_t)$. Note that $C = \sum_i C_i$. For each singular point of C, we shall need to consider the mixed multiplier ideal $\mathcal{J}(\boldsymbol{x} \cdot \boldsymbol{C}) = \mathcal{J}(x^1 C_1 + \ldots x^t C_t)$ with $\boldsymbol{x}$ varying in the hypercube $[0, 1)^t$. Interpreting the results of § 2 in this context, it follows that the jumping walls associated to these multiplier ideals cut up the hypercube into convex rational polytopes on which the map $x \mapsto \mathcal{J}(\boldsymbol{x} \cdot \boldsymbol{C})$ is constant. Note that the fibres of this map are neither open nor closed.

Definition 3.3. A *face*[1] associated to C endowed with the partition $\boldsymbol{C}$ is a finite intersection of jumping walls and coordinate hyperplanes.

Remark 3.4. Even if we are interested in the jumping walls intersecting the hypercube, the context of rational divisors is not sufficient, since the jumping walls are determined by relevant values associated to ideal sheaves—more precisely, a jumping wall might intersect a coordinate axis in a jumping number bigger than 1 associated only to an ideal sheaf.

[1] In a previous version of the paper these faces were called walls and A. Libgober kindly pointed to me that using the word *wall* was misleading in codimension ≥ 2.

If W is a face associated to a curve C endowed with the partition $\boldsymbol{C}$, the set $\mathcal{U}(W)$ is the set of the connected components of the difference between W and the union of all the jumping walls and coordinate hyperplanes that do not contain W. The mixed multiplier ideal is constant on each $U \in \mathcal{U}(W)$ and will be denoted by $\mathcal{J}(U \cdot \boldsymbol{C})$. Furthermore, if $\boldsymbol{d}$ is the vector $(d_1, \dots, d_t)$, where $\deg C_i = d_i$, we define the height function $h_{\boldsymbol{C}} : \mathbb{R}^t \to \mathbb{R}$ by $h_{\boldsymbol{C}}(\boldsymbol{x}) = \boldsymbol{d} \cdot \boldsymbol{x}$.

Definition 3.5. A face W of the projective curve C endowed with the partition $\boldsymbol{C}$ is called a *distinguished face* if the height function $h_{\boldsymbol{C}}$ is constant on W. The set of distinguished faces will be denoted by $\mathfrak{F}(\boldsymbol{C})$.

LEMMA 3.6. *If $C \subset \mathbb{P}^2$ is endowed with the partition $\boldsymbol{C}$, C_i is a component in $\boldsymbol{C}$ and W a distinguished face, then either $x^i = 0$ along W, or C_i passes through P, a singular point of C, to which one of the jumping walls that cut out W is associated.*

Proof. It is easy to see that a distinguished face W, seen as a subset in the first orthant of $\mathbb{R}^t$, is bounded for the euclidean metric. Suppose that $W \not\subset \{x^i = 0\}$. Then the component C_i must satisfy the conclusion since otherwise the corresponding coordinate x^i would be unbounded. □

EXAMPLE 3.7. Let $C_1, \dots, C_6$ be the lines of Ceva's arrangement $C = \sum C_i$; C_i and C_j intersect in a node of the arrangement if and only if $i+j = 7$. Ceva's arrangement has four triple points: $C_4 \cap C_5 \cap C_6$, $C_1 \cap C_2 \cap C_4$, $C_2 \cap C_3 \cap C_6$ and $C_1 \cap C_3 \cap C_5$. For C endowed with the partition $\boldsymbol{C} = (C_1, \dots, C_6)$ there are five distinguished faces: one for each triple point and one for all four. Clearly for each triple point P there is a distinguished face W_P; for example if $P = C_4 \cap C_5 \cap C_6$ then W_P is defined by $x^4 + x^5 + x^6 = 2$, $x^1 = x^2 = x^3 = 0$ and $h_{\boldsymbol{C}}(W_P) = 2$. Now, if W is a distinguished face different from the W_P, then let $\varphi_\alpha(\boldsymbol{x}) = 2$ be the equations defining the jumping walls that cut out W—2 is the only relevant value. Note that each equation is of the form $x^i + x^j + x^k = 2$. Let $I \subset \{1, 2, \dots, 6\}$ be the set of subscripts appearing in the equations φ_α. Since W is distinguished, $x^j = 0$ along W for every $j \notin I$. Furthermore the equation

$$\sum_{i \in I} x^i = h_{\boldsymbol{C}}(W)$$

is a linear combination of the φ_α. Hence there exist ζ_α such that

$$\sum_\alpha \zeta_\alpha(\varphi_\alpha(\boldsymbol{x}) - 2) = \sum_{i \in I} x^i - h_{\boldsymbol{C}}(W)$$

for any $\boldsymbol{x} \in \mathbb{R}^6$. Hence $2\sum_\alpha \zeta_\alpha = h_{\boldsymbol{C}}(W)$, and taking $x^i = 1$ for every $i \in I$, $3\sum_\alpha \zeta_\alpha = |I|$. It follows that $2|I| = 3h_{\boldsymbol{C}}(W)$, *i.e.* that $|I| = 6$ and $h_{\boldsymbol{C}}(W) = 4$. To see that W is unique with these properties it is sufficient to notice that W is defined by the four equations corresponding to the four triple points. It is clear that it should be defined by at least three out of four equations. Summing these three equations and using $\sum_1^6 x^i = 4$ we get the fourth.

Example 3.8. Let Γ_1 and Γ_2 be two conics that have common tangents at the two points of intersection P and Q. Let H_∞ be the line through P and Q. We want to determine the set of distinguished faces $\mathfrak{F}_d$ for the curve $C = C_1 + C_2$, with the partition $\boldsymbol{C} = \{C_1, C_2\}$, where $C_1 = \Gamma_1 + \Gamma_2$ and $C_2 = H_\infty$. The curve C_1 has two tacnodes at P and Q; a jumping number $3/4$ and hence a unique relevant value 3. The curve $C = C_1 + C_2$ has two singular points and the exceptional configuration of the minimal log-resolution is $(2+1)E_1 + (4+1)E_2$. There are two jumping values, $3/5$ and $4/5$ and two relevant values 3 and 4 associated to the second exceptional divisor in the log-resolution for each singular point. It follows that there are two jumping walls W_3 and W_4 defined by $4x^1 + x^2 = 3$ and $4x^1 + x^2 = 4$ respectively. There are three faces and all three are distinguished since $h_{\boldsymbol{C}} = 4x^1 + x^2$: W_3, W_4 and the intersection of W_3 with the coordinate line $\{x^2 = 0\}$, *i.e.* the point W_0 of coordinates $(3/4, 0)$. Finally,

$$\mathcal{U}(W_0) = \{W_0\}, \quad \mathcal{U}(W_3) = \{W_3 \setminus W_0\} \quad \text{and} \quad \mathcal{U}(W_4) = \{W_4\}.$$

3.3 The irregularity

In this section we state and prove the formula for the irregularity of the abelian covering $S' = S(\boldsymbol{n}, M, \boldsymbol{C}, H_\infty)$—the standard $\oplus_{j=1}^{s}\mathbb{Z}/n_j\mathbb{Z}$-covering $S' \to \mathbb{P}^2$ defined by the linear equivalences

$$n_j L_{\chi_j} \sim \sum_{i=1}^{t} \mu_j^i C_i + \left(\left\lceil \frac{1}{n_j} \sum_{i=1}^{t} \mu_j^i d_i \right\rceil n_j - \sum_{i=1}^{t} \mu_j^i d_i \right) H_\infty,$$

where $\mathcal{L}_{\chi_j} = \mathcal{O}_{\mathbb{P}^2}(\left\lceil \sum_{i=1}^{t} \mu_j^i d_i / n_j \right\rceil)$, $d_i = \deg C_i$, $\boldsymbol{C} = (C_1, \dots, C_t)$, $\boldsymbol{n} = (n_1, \dots, n_s)$ and M denotes the $t \times s$ matrix $[\mu_j^i]$ (see Notation 3.1).

For any rational convex polytope $U \subset \mathbb{R}^t$ set

$$|U|_{\boldsymbol{n}}^{M} = \operatorname{card} \varphi^{-1}(U \cap [0,1)^t),$$

where the map $\varphi : [0,1)^s \cap \bigoplus_{j=1}^{s} 1/n_j\mathbb{Z} \longrightarrow [0,1)^t$, depending on $\boldsymbol{n} = (n_1, \dots, n_s)$ and the matrix M, is defined by

$$\varphi\left(\frac{a^1}{n_1}, \dots, \frac{a^s}{n_s}\right) = \left(\left\langle \sum_j \mu_j^1 \frac{a^j}{n_j} \right\rangle, \dots, \left\langle \sum_j \mu_j^t \frac{a^j}{n_j} \right\rangle \right).$$

In case M is the identity matrix we shall omit the superscript M it the notation $|U|_{\boldsymbol{n}}^{M}$. Similarly, if $n_j = n$ for every j, we shall use $|U|_n^M$ for $|U|_{(n,\dots,n)}^{M}$. Note that $|U|_n$ is the number of rational points in W, points whose coordinates belong to $1/n\mathbb{Z}$ and are non-negative and smaller than 1.

Theorem 3.9. *Let S be the normalization of the $\oplus_{j=1}^{s}\mathbb{Z}/n_j\mathbb{Z}$-abelian covering $S' \to \mathbb{P}^2$ with $S' = S(\boldsymbol{n}, M, \boldsymbol{C}, H_\infty)$. Suppose that S is a connected surface. Then*

$$q(S) = \sum_{W \in \mathfrak{F}(\boldsymbol{C})} \sum_{U \in \mathcal{U}(W)} |U|_{\boldsymbol{n}}^{M} \cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + h_{\boldsymbol{C}}(W)) \otimes \mathcal{J}(U \cdot \boldsymbol{C})\big), \qquad (7)$$

if $\sum_i C_i$ is transverse to H_∞, and

$$q(S) = \sum_{W \in \mathfrak{F}(\overline{\boldsymbol{C}})} \sum_{U \in \mathcal{U}(W)} |U|_{\boldsymbol{n}}^{\overline{M}} \cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + h_{\overline{\boldsymbol{C}}}(W)) \otimes \mathcal{J}(U \cdot \overline{\boldsymbol{C}})\big), \tag{8}$$

if $\sum_i C_i$ is not transverse to H_∞ and the covering is branched along H_∞, where $\overline{\boldsymbol{C}} = (H_\infty, \boldsymbol{C})$ and

$$\overline{M} = \begin{bmatrix} \mu_1^0 & \cdots & \mu_s^0 \\ & M & \end{bmatrix}$$

with $\mu_j^0 = \left\lceil \sum_{i=1}^t \mu_j^i d_i / n_j \right\rceil n_j - \sum_{i=1}^t \mu_j^i d_i$.

Proof. In order to compute the irregularity of S we need to see S as a standard abelian covering of group $\oplus_{j=1}^s \mathbb{Z}/n_j\mathbb{Z}$. We use the normalization algorithm from [19]. Let $\mu : X \to \mathbb{P}^2$ be a log resolution of the branch divisor. According to the position of the line at infinity, the points that are blown up lie either on $\sum_i C_i$ or on $\sum_i C_i + H_\infty$. The abelian covering $S' \to \mathbb{P}^2$ pulls back to a standard abelian covering $S'' \to X$ defined by line bundles $\mathcal{L}''_{\chi_j}$. Then, the normalization procedure yields the normal surface S with only Hirzebruch-Jung singularities.

$$\begin{array}{ccccc} S & \longrightarrow & S'' & \longrightarrow & S' \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow & & \Big\downarrow \\ X & =\!= & X & \xrightarrow{\ \mu\ } & \mathbb{P}^2 \end{array}$$

It is a standard abelian covering with line bundles $\mathcal{L}_{\chi_j}$ among the elements of the reduced building data. Using the Leray spectral sequence and the Serre duality,

$$q(S) = h^1(S, \mathcal{O}_S) = h^1(X, \pi_* \mathcal{O}_S) = \sum_{\chi \in \widehat{G}} h^1(X, \omega_X \otimes \mathcal{L}_\chi).$$

For the computation of the terms in the right hand member, we distinguish two cases.

First case. H_∞ is transverse to $\sum_i C_i$. Let

$$\mu^* C_i = \widetilde{C}_i + \sum_P \boldsymbol{e}_i^P \cdot \boldsymbol{E}_P \tag{9}$$

the sum being taken over all the singular points of $\sum_i C_i$ excepting the nodes. Here and in the sequel $\boldsymbol{e}_i^P \cdot \boldsymbol{E}_P$ denotes the sum

$$\sum_\alpha e_i^{P,\alpha} E_{P,\alpha}$$

where $E_{P,\alpha}$ are the irreducible components of the exceptional configuration of the log resolution μ over P. The line bundles $L''_{\chi_j} \sim \left\lceil \sum_i \mu_j^i d_i / n_j \right\rceil \widetilde{H}$ and the linear equivalences

$$n_j L''_{\chi_j} \sim \sum_i \mu_j^i \widetilde{C}_i + \sum_{i,P} \mu_j^i \boldsymbol{e}_i^P \cdot \boldsymbol{E}_P + \left(\left\lceil \sum_i \mu_j^i d_i / n_j \right\rceil n_j - \sum_i \mu_j^i d_i \right) \widetilde{H}_\infty,$$

holding for each $1 \leq j \leq s$, define S''. If $\chi = \chi_1^{a^1} \cdots \chi_s^{a^s}$, $0 \leq a^j < n_j$, then, after normalization, by [17, Proposition 3.2] and by Proposition A.1, L_χ is linearly equivalent to

$$\sum_j a^j L''_{\chi_j} - \sum_i \left\lfloor \sum_j \frac{a^j \mu_j^i}{n_j} \right\rfloor \widetilde{C}_i - \sum_P \left\lfloor \sum_{i,j} \frac{a^j \mu_j^i}{n_j} \boldsymbol{e}_i^P \right\rfloor \cdot \boldsymbol{E}_P$$
$$- \left\lfloor \sum_{j=1}^s \frac{a^j}{n_j} \Big(\left\lceil \sum_i \mu_j^i d_i / n_j \right\rceil n_j - \sum_i \mu_j^i d_i \Big) \right\rfloor \widetilde{H}_\infty,$$

and using (9) and $\widetilde{H}_\infty \sim \widetilde{H}$ to

$$\left(\left\lceil \sum_{i,j} \frac{a^j \mu_j^i d_i}{n_j} \right\rceil - \sum_i \left\lfloor \sum_j \frac{a^j \mu_j^i}{n_j} \right\rfloor d_i \right) \widetilde{H} - \left(\sum_P \left\lfloor \sum_{i,j} \frac{a^j \mu_j^i}{n_j} \boldsymbol{e}_i^P \right\rfloor + \sum_{P,i} \left\lfloor \sum_j \frac{a^j \mu_j^i}{n_j} \right\rfloor \boldsymbol{e}_i^P \right) \cdot \boldsymbol{E}_P.$$

Setting

$$x^i = \left\langle \sum_j a^j \mu_j^i / n_j \right\rangle \tag{10}$$

for every $1 \leq i \leq t$, it follows that

$$L_\chi \sim \left\lceil \sum_i d_i x^i \right\rceil \widetilde{H} - \sum_P \left\lfloor \sum_i x^i \boldsymbol{e}_i^P \right\rfloor \cdot \boldsymbol{E}_P.$$

Then

$$h^1(X, K_X + L_\chi) = h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + \left\lceil \sum_i d_i x^i \right\rceil) \otimes \mathcal{J}(\sum_i x^i C_i)) \tag{11}$$

and the dimension $h^1(X, K_X + L_\chi)$ might be non-zero whenever the numbers x^i satisfy three conditions. First $\sum_i x^i d_i$ must be an integer. If not, then the right-hand side of (11) vanishes by the Kawamata-Viehweg-Nadel vanishing theorem. Second, for every curve C_i there exists a singular point P of B lying on C_i and a relevant position α of P such that the number $\sum_i x^i e_i^{P,\alpha}$ is a relevant value of D at (P, α). Indeed, if this condition does not hold for C_1 for example, it is sufficient to notice that $\mathcal{J}(D') = \mathcal{J}(\sum_i x^i C_i)$, where $D' = (x^1 - \varepsilon)C_1 + \sum_{i=2}^t x^i C_i$ for $\varepsilon > 0$ sufficiently small, and to apply the Kawamata-Viehweg-Nadel vanishing theorem to see that the right-hand side of (11) vanishes. Third, suppose that $\sum_i x^i d_i$ is an integer and that for every component C_i there exists a singular point P of B and a position α such that $r_{P,\alpha} = \sum_i x^i e_i^{P,\alpha}$ is a relevant value of D at (P, α). If W is the space of solutions of these equations seen as equations in the unknowns x^i, then W is a face for the partition $\boldsymbol{C} = (C_1, \ldots, C_t)$ and the linear operator $h_{\boldsymbol{C}} : \boldsymbol{x} \mapsto \sum_i x^i d_i$ must be constant on W. Indeed, if W is positive dimensional and not contained into a fibre of $h_{\boldsymbol{C}}$, it is sufficient to take $\boldsymbol{y} \in W$ such that $\sum_i y^i d_i < \sum_i x^i d_i$. Then, if $\delta = \lceil \sum_i y^i d_i \rceil = \sum_i x^i d_i$,

$$h^1(\mathbb{P}^2, \mathcal{O}_X(-3 + \sum_i d_i x^i) \otimes \mathcal{J}(\sum_i x^i C_i)) = h^1(\mathbb{P}^2, \mathcal{O}_X(-3 + \delta) \otimes \mathcal{J}(\sum_i y^i C_i)) = 0.$$

So the face W is distinguished and $\mathcal{J}(\sum_i x^i C_i) = \mathcal{J}(U \cdot \boldsymbol{C})$, for U the corresponding connected components defined on W by the other jumping walls and coordinate hyperplanes.

By the previous considerations we conclude that

$$\begin{aligned} q(S) &= \sum_{\chi\in\widehat{G}} h^1(X, K_X+L_\chi) \\ &= \sum_{W\in\mathfrak{F}}\sum_{U\in\mathcal{U}(W)} |U|^M_{\boldsymbol{n}}\cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3+h_{\boldsymbol{C}}(W))\otimes\mathcal{J}(U\cdot\boldsymbol{C})\big), \end{aligned}$$

where

$$|U|^M_{\boldsymbol{n}} = \operatorname{card}\left\{(a^1,\ldots,a^s)\,\Big|\, 0\le a^j<n_j, \left(\left\langle\sum_j a^j\mu^1_j/n_j\right\rangle,\ldots,\left\langle\sum_j a^j\mu^t_j/n_j\right\rangle\right)\in U\right\}.$$

Second case. If H_∞ is not transverse to $\sum_i C_i$ and S' is ramified above H_∞, then, supposing for simplicity that there is only one singular point Q of $\sum_i C_i$ lying on H_∞, we have

$$\mu^*(\sum_i C_i + H_\infty) = \sum_i \widetilde{C}_i + \widetilde{H}_\infty + \sum_{P\neq Q}\boldsymbol{e}^P\cdot\boldsymbol{E}_P + (\boldsymbol{e}_Q+\boldsymbol{e}^Q_\infty)\cdot\boldsymbol{E}_Q,$$

with $\boldsymbol{e}^P=\sum_i\boldsymbol{e}^P_i$ and $\boldsymbol{e}^Q_\infty=(1,\ldots)$. As in the first case,

$$\begin{aligned} n_jL''_{\chi_j} &\sim \sum_i\mu^i_j\widetilde{C}_i + \sum_{i,P}\mu^i_j\boldsymbol{e}^P_i\cdot\boldsymbol{E}_P \\ &+\left(\left\lceil\sum_i\mu^i_jd_i/n_j\right\rceil n_j - \sum_i\mu^i_jd_i\right)\widetilde{H}_\infty + \left(\left\lceil\sum_i\mu^i_jd_i/n_j\right\rceil n_j - \sum_i\mu^i_jd_i\right)\boldsymbol{e}^Q_\infty\cdot\boldsymbol{E}_Q, \end{aligned}$$

hence

$$\begin{aligned} L''_\chi &\sim \sum_j a^jL''_{\chi_j} - \sum_i\left\lfloor\sum_j\frac{a^j\mu^i_j}{n_j}\right\rfloor\widetilde{C}_i - \left\lfloor\sum_{j=1}^s\frac{a^j}{n_j}\Big(\left\lceil\sum_i\mu^i_jd_i/n_j\right\rceil n_j - \sum_i\mu^i_jd_i\Big)\right\rfloor\widetilde{H}_\infty \\ &-\sum_{P\neq Q}\left\lfloor\sum_{i,j}\frac{a^j\mu^i_j}{n_j}\boldsymbol{e}^P_i\right\rfloor\cdot\boldsymbol{E}_P - \left\lfloor\sum_{i,j}\frac{a^j\mu^i_j}{n_j}\boldsymbol{e}^Q_i + \sum_{j=1}^s\frac{a^j}{n_j}\Big(\left\lceil\sum_i\mu^i_jd_i/n_j\right\rceil n_j - \sum_i\mu^i_jd_i\Big)\boldsymbol{e}^Q_\infty\right\rfloor\cdot\boldsymbol{E}_Q. \end{aligned}$$

By (9) and $\widetilde{H}_\infty\sim\widetilde{H}-\boldsymbol{e}^Q_\infty\cdot\boldsymbol{E}_Q$,

$$\begin{aligned} L_\chi &\sim \left(\left\lceil\sum_{i,j}\frac{a^j\mu^i_jd_i}{n_j}\right\rceil - \sum_i\left\lfloor\sum_j\frac{a^j\mu^i_j}{n_j}\right\rfloor d_i\right)\widetilde{H} \\ &-\sum_{P\neq Q}\left(\left\lfloor\sum_{i,j}\frac{a^j\mu^i_j}{n_j}\boldsymbol{e}^P_i\right\rfloor - \sum_{P,i}\left\lfloor\sum_j\frac{a^j\mu^i_j}{n_j}\right\rfloor\boldsymbol{e}^P_i\right)\cdot\boldsymbol{E}_P \\ &-\left(\left\lfloor\sum_{i,j}\frac{a^j\mu^i_j}{n_j}\boldsymbol{e}^Q_i - \sum_{i,j}\frac{a^j\mu^i_jd_i}{n_j}\boldsymbol{e}^Q_\infty\right\rfloor - \sum_i\left\lfloor\sum_j\frac{a^j\mu^i_j}{n_j}\right\rfloor\boldsymbol{e}^Q_i + \left\lceil\sum_{i,j}\frac{a^j\mu^i_jd_i}{n_j}\right\rceil\boldsymbol{e}^Q_\infty\right)\cdot\boldsymbol{E}_Q. \end{aligned}$$

Set $x^i = \left\langle \sum_j a^j \mu_j^i / n_j \right\rangle$, $1 \le i \le t$, as in (10). Then

$$L_\chi \sim \left\lceil \sum_{i=1}^{t} d_i x^i \right\rceil \widetilde{H} - \sum_{P \neq Q} \left\lfloor \sum_{i=1}^{t} x^i \boldsymbol{e}_i^P \right\rfloor \cdot \boldsymbol{E}_P$$
$$- \left(\left\lfloor \sum_i x^i \boldsymbol{e}_i^Q - \sum_i d_i x^i \boldsymbol{e}_\infty^Q \right\rfloor + \left\lceil \sum_i d_i x^i \right\rceil \boldsymbol{e}_\infty^Q \right) \cdot \boldsymbol{E}_Q. \quad (12)$$

In the formula for the irregularity, two things may happen. Either $\sum_i d_i x^i$ is an integer and the superabundances involved can be dealt with as before, or $\sum_i d_i x^i$ is not an integer. In this latter situation set $C_0 = H_\infty$, $d_0 = 1$, $\boldsymbol{e}_0^P = \mathbf{0}$ if $P \neq Q$ and

$$x^0 = \left\lceil \sum_i d_i x^i \right\rceil - \sum_i d^i x_i.$$

Then

$$\begin{aligned} L_\chi &\sim \left(x^0 + \sum_{i=1}^{t} d_i x^i \right) \widetilde{H} - \sum_{P \neq Q} \left\lfloor \sum_{i=1}^{t} x^i \boldsymbol{e}_i^P \right\rfloor \cdot \boldsymbol{E}_P - \left\lfloor x^0 \boldsymbol{e}_0^Q + \sum_{i=1}^{t} x^i \boldsymbol{e}_i^Q \right\rfloor \cdot \boldsymbol{E}_Q \\ &= \left(\sum_{i=0}^{t} d_i x^i \right) \widetilde{H} - \sum_{P} \left\lfloor \sum_{i=0}^{t} x^i \boldsymbol{e}_i^P \right\rfloor \cdot \boldsymbol{E}_P \end{aligned}$$

and the formula for the irregularity follows as before replacing $\sum_{i=1}^t C_i$ by $\sum_{i=0}^t C_i$, $\boldsymbol{C}$ by $\overline{\boldsymbol{C}} = (C_0, \dots, C_t)$ and M by $\overline{M}$, where

$$\overline{M} = \begin{bmatrix} \mu_1^0 & \cdots & \mu_s^0 \\ & M & \end{bmatrix}$$

with $\mu_j^0 = \left\lceil \sum_{i=1}^t \mu_j^i d_i / n_j \right\rceil n_j - \sum_{i=1}^t \mu_j^i d_i$. To end the proof it remains to show that

$$x^0 = \left\langle \sum_j \frac{a^j \mu_j^0}{n_j} \right\rangle.$$

But this is clear, since

$$\begin{aligned} \left\langle \sum_j \frac{a^j \mu_j^0}{n_j} \right\rangle &= \left\langle \sum_{j=1}^{s} \left(\left\lceil \frac{1}{n_j} \sum_{i=1}^{t} \mu_j^i d_i \right\rceil n_j - \sum_{i=1}^{t} \mu_j^i d_i \right) \frac{a^j}{n_j} \right\rangle \\ &= \left\langle \left\lceil \sum_{i,j} d_i \frac{a^j \mu_j^i}{n_j} \right\rceil - \sum_{i,j} d_i \frac{a^j \mu_j^i}{n_j} \right\rangle \\ &= \left\lceil \sum_{i=1}^{t} d_i \left\langle \sum_{j=1}^{s} \frac{a^j \mu_j^i}{n_j} \right\rangle \right\rceil - \sum_{i=1}^{t} d_i \left\langle \sum_{j=1}^{s} \frac{a^j \mu_j^i}{n_j} \right\rangle \end{aligned}$$

and this equals x^0 by the definition of the x^i when $1 \le i \le t$. $\square$

4 Applications and examples

4.1 Asymptotic behaviour of the irregularity

In setting out to look for applications of Theorem 3.9 it seems best to start with the asymptotic behaviour of the irregularity of the abelian coverings of the projective plane described by E. Hironaka in [6].

Corollary 4.1. *Let $S' \to \mathbb{P}^2$ be the $(\mathbb{Z}/n\mathbb{Z})^s$-abelian covering defined by $L_{\chi_j} \sim C_j + (\lceil d_j/n \rceil n - d_j)H_\infty$, $1 \le j \le s$, with $d_j = \deg C_j$. If S is the normalization of S', then*

$$q(S) = \sum_{W \in \mathfrak{F}(\boldsymbol{C})} \sum_{U \in \mathcal{U}(W)} |U|_n \cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + h_{\boldsymbol{C}}(W)) \otimes \mathcal{J}(U \cdot \boldsymbol{C})\big)$$

where the partition $\boldsymbol{C}$ is given either by the curves $C_1, \ldots, C_s$ or by the curves H_∞, $C_1, \ldots, C_s$, depending on whether or not H_∞ is transverse to $\sum_i C_i$.

Proof. By Theorem 3.9, when $\sum_i C_i$ is not transverse to H_∞ and the covering is branched along H_∞,

$$q(S) = \sum_{W \in \mathfrak{F}(\overline{\boldsymbol{C}})} \sum_{U \in \mathcal{U}(W)} |U|_n^{\overline{M}} \cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + h_{\overline{\boldsymbol{C}}}(W)) \otimes \mathcal{J}(U \cdot \overline{\boldsymbol{C}})\big), \tag{13}$$

where $\overline{\boldsymbol{C}} = (H_\infty, \boldsymbol{C})$ and $\overline{M} = \begin{bmatrix} \mu_1^0 & \cdots & \mu_s^0 \\ & I_s & \end{bmatrix}$, with $\mu_j^0 = \lceil d_j/n \rceil n - d_j$ for $1 \le j \le s$. Moreover, $|U|_n^{\overline{M}} = \operatorname{card} \varphi^{-1}(U \cap [0,1)^{s+1})$, where the map $\varphi : [0,1)^s \cap (1/n\mathbb{Z})^s \to [0,1)^{s+1}$ associated to $\overline{M}$ is defined by

$$\varphi\left(\frac{a^1}{n_1}, \ldots, \frac{a^s}{n_s}\right) = \left(\left\langle \sum_j \mu_j^0 \frac{a^j}{n_j} \right\rangle, \frac{a^1}{n}, \ldots, \frac{a^s}{n}\right).$$

Since φ is injective, it follows that

$$|U|_n^{\overline{M}} = \operatorname{card}(U \cap [0,1)^{s+1} \cap (1/n\mathbb{Z})^{s+1}) = |U|_n.$$

□

Using Corollary 4.1 we can recover E. Hironaka's result concerning the asymptotic behaviour of the irregularity of the abelian covering Σ_n. See also [1, Theorem 1.7], where N. Budur establish the quasi-polynomial behaviour of the Hodge numbers $h^{0,q}$ of the finite abelian coverings of a smooth n-dimensional variety.

Corollary 4.2. *Let $C \subset \mathbb{P}^2$ be a reduced curve and $\Omega = \mathbb{P}^2 \smallsetminus (C \cup H_\infty)$. Let Σ_n be the unique normal abelian covering associated to the natural epimorphism*

$$\pi_1(\Omega) \longrightarrow H_1(\Omega, \mathbb{Z}) \longrightarrow H_1(\Omega, \mathbb{Z}/n\mathbb{Z}) \simeq (\mathbb{Z}/n\mathbb{Z})^s,$$

with s the number of connected components of C. Then $q(\Sigma_n)$ is a quasi-polynomial function of n of degree $\le s$.

Definition. A function $f : \mathbb{N} \to \mathbb{N}$ is called a quasi-polynomial function if there exists an integer $N > 0$ and polynomials $P_0, \ldots, P_{N-1}$ such that $f(n) = P_j(N)$ if $n \equiv j \mod N$.

Proof. By Remark 3.2, the surface Σ_n coincides with the normalization of the abelian covering of the projective plane with group $(\mathbb{Z}/n\mathbb{Z})^s$, associated to $C = \sum_{j=1}^{s} C_j$ and H_∞ and determined by

$$nL_j \sim C_j + (\lceil d_j/n \rceil\, n - d_j)H_\infty.$$

By Corollary 4.1 we have

$$q(\Sigma_n) = \sum_{W \in \mathfrak{F}(\boldsymbol{C})} \sum_{U \in \mathcal{U}(W)} |U|_n \cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + h_{\boldsymbol{C}}(W)) \otimes \mathcal{J}(U \cdot \boldsymbol{C})\big),$$

with $\boldsymbol{C}$ the partition of C induced by the curves C_j. The closure of the subset $U \subset W$ in W represents a convex polytope and its border in W a finite union of convex polytopes. Now, if $\mathcal{P} \subset \mathbb{R}^s$ is a convex polytope, the *Ehrhart quasi-polynomial* of $\mathcal{P}$ is the function defined by

$$i(\mathcal{P}, n) = \operatorname{card}(n\mathcal{P} \cap \mathbb{Z}^s),$$

where $n\mathcal{P} = \{n\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{P}\}$. Clearly, the number $i(\mathcal{P}, n)$ is equal to the number of rational points in $\mathcal{P} \cap (1/n\mathbb{Z})^s$. We refer the reader to [21, Theorem 4.6.25] where it is shown that $i(\mathcal{P}, n)$ is indeed a quasi-polynomial whose degree is $\dim \mathcal{P}$. The result follows. □

4.2 Cyclic coverings

As a particular case of Theorem 3.9 we obtain the formula for the irregularity of cyclic multiple planes. This study has been initiated by O. Zariski in [23] where he computed the irregularity in case the branching curve has only nodes and cusps as singularities. Various generalizations have since been proposed to Zariski's formula in [3, 12, 13, 16, 22, 17].

Corollary 4.3. *Let $C \subset \mathbb{P}^2$ be a curve of degree d and H_∞ a line transverse to C. If S_n is the normalization of the standard cyclic $\mathbb{Z}/n\mathbb{Z}$-covering of the plane defined by the linear equivalence $nL_\chi \sim C + (\lceil d/n \rceil\, n - d)H_\infty$, then*

$$q(S_n) = \sum_{\substack{\xi \text{ jumping number of } C \\ \xi \in 1/(n \wedge d)\,\mathbb{Z},\ 0<\xi<1}} h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + \xi d) \otimes \mathcal{J}(\xi \cdot C)).$$

Proof. Since H_∞ is transverse to C, then the faces in the formula (7) are points ξ in the open interval $(0, 1)$ corresponding to the jumping numbers of C such that $\xi \cdot \deg C \in \mathbb{Z}$. Moreover $|\xi|_n = \operatorname{card}(\{\xi\} \cap 1/n\mathbb{Z})$ equals 1 or 0 depending on weather $\xi n \in \mathbb{Z}$. The result follows. □

In the case of a cyclic covering, if H_∞ is not transverse to C, then the faces in the formula (13) live in $\mathbb{R}^2$ with euclidean coordinates x and x^∞. They are of two types: 1) those for which $x^\infty = 0$, in which case they are jumping numbers for C and the corresponding term in (7) is determined as in the above corollary; 2) those for which $x^\infty \neq 0$ and the face is determined by an equation whose homogeneous part must coincide with the linear form $h_{(H_\infty, C)}$ modulo the multiplication by a non-zero rational. It may be said that the results obtained for the irregularity are qualitatively different. In the transverse situation the irregularity is constant as a function of n. In the non transverse situation the irregularity depends on n. More precisely, using Hironaka's result, it is a quasi-polynomial of degree ≤ 1. The next example illustrates this behaviour of the irregularity in the non transverse situation.

EXAMPLE 4.4. The two ellipses Γ_i, $i = 1, 2$, with common tangents at P and Q considered in Example 3.8 provide cyclic coverings with maximal degrees for the quasi-polynomials that represent the irregularity, whatever the relative position of the line at infinity.

If H_∞ is transverse to $B = \Gamma_1 + \Gamma_2$, then the $\mathbb{Z}/n\mathbb{Z}$-cyclic covering S_n has non vanishing irregularity if and only if n is divisible by 4, since $3/4$ is the only jumping number of B, in which case

$$q(S_n) = h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) = 1.$$

Now, if H_∞ is the line through P and Q, then we have seen in Example 3.8 that there are two jumping walls W_3 and W_4, and three distinguished faces, the previous two and the point W_0, the intersection of W_3 with the coordinate plane $x^\infty = 0$. Set $C = \Gamma_1 + \Gamma_2 + H_\infty$. By Corollary 4.1, we get, for $n \geq 5$[2],

$$\begin{aligned} q(S_n) &= |W_0|_n \, h^1(\mathbb{P}^2, \mathcal{J}(W_0 \cdot C)) + |W_3 \smallsetminus W_0|_n \, h^1(\mathbb{P}^2, \mathcal{J}(W_3 \smallsetminus W_0 \cdot C)) \\ &\quad + |W_4|_n \, h^1(\mathbb{P}^2, \mathcal{O}(1) \otimes \mathcal{J}(W_4 \cdot C)) \\ &= |W_0|_n \, h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) + |W_3 \smallsetminus W_0|_n \, h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) + |W_4|_n \, h^1(\mathbb{P}^2, \mathcal{I}_Z(1)) \\ &= |W_3|_n \, h^1(\mathbb{P}^2, \mathcal{I}_{P,Q}) + |W_4|_n \, h^1(\mathbb{P}^2, \mathcal{I}_Z(1)), \end{aligned}$$

where Z is the subscheme supported at P and Q and determined by the points and the directions of the tangents to the two conics at P and Q. Since

$$|W_l|_n = \operatorname{card}\{(x, x^\infty) \mid 4x + x^\infty = l,\ 0 \leq x < 1,\ 0 \leq x^\infty < 1, x, x^\infty \in 1/n\mathbb{Z}\},$$

$l = 3, 4$, it follows that

$$q(S_n) = \left\lfloor \frac{n+1}{4} \right\rfloor + \left\lfloor \frac{n+3}{4} \right\rfloor.$$

EXAMPLE 4.5. If in the previous example we consider the abelian covering Σ_n of $\mathbb{P}^2$ with group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and branched along $C = \Gamma_1 + \Gamma_2 + H_\infty$ with the partition $\boldsymbol{C} = (\Gamma_1, \Gamma_2, H_\infty)$, then the formula for the irregularity is the same but the faces are defined in $\mathbb{R}^3$ by

$$W_l = \{(x^1, x^2, x^\infty) \mid 2x^1 + 2x^2 + x^\infty = l\},$$

$l = 3, 4$. Then

$$|W_3|_n = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left(n + \left\lceil \frac{n}{2} \right\rceil - 3 \right) \quad \text{and} \quad |W_4|_n = \frac{1}{2} \left\lfloor \frac{n-1}{2} \right\rfloor \left(\left\lfloor \frac{n-1}{2} \right\rfloor - 1 \right),$$

hence $q(\Sigma_n) = (n-1)(n-2)/2$.

[2]For $n = 4$ the covering is branched only along the two conics and the irregularity equals 1. For $n = 3$, the formula for the non transverse intersection applies and the irregularity equals 2. For $n = 2$ the covering is branched again only along the conics and $q = 0$.

4.3 Line arrangements with only triple points

Next we want to point out that the formula (7) simplifies in case the branching curve is a line arrangement $\mathcal{A}$ with only triple points.

NOTATION. Let W be a face. The subarrangement $\mathcal{A}_W$ will denote the minimal subarrangement of $\mathcal{A}$ determined by the points that contribute to W. This subarrangement is unique since all points are triple points.

THEOREM 4.6. *Let $\mathcal{A} = \bigcup_{j=1}^{m} H_j$ be a line arrangement in the projective plane and let H_∞ be a line either of $\mathcal{A}$ or transverse to $\mathcal{A}$. Let $s = m-1$ in the former case and $s = m$ in the latter. If S is the normalization of the standard abelian covering associated to $\mathcal{A}$, the line H_∞ and the group $G \simeq (\mathbb{Z}/n\mathbb{Z})^s$, then*

$$q(S) = \sum_{W \in \mathfrak{F}(\mathcal{A})} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + 2/3 \deg \mathcal{A}_W) \otimes \mathcal{I}_{Z_U}\big). \tag{14}$$

Proof. For each singular point P of the arrangement, the configuration of exceptional divisors is reduced to only one divisor E_P, with $e_P^{\mathcal{A}} = 3$. Moreover, $e_j^P = 1$ or 0 depending on weather the line H_j passes through P or not. Since $2/3$ is the only jumping number smaller than 1 for a triple point, the only relevant value is 2. It follows that for any P, the elementary wall W_P is given by

$$W_P = \{(x^1, \ldots, x^s) \mid e_P^1 x^1 + \ldots + e_P^s x^s = 2\}.$$

Now, let W be a bounded face in the formula for the irregularity. There exists a unique minimal subarrangement $\mathcal{A}_W$ determined by the points contributing to W.

Claim. $h_{\mathcal{A}}(W) = 2/3 \deg(\mathcal{A}_W)$.
Indeed, let $I \subset \{1, \ldots, s\}$ such that $\mathcal{A}_W = \bigcup_{i \in I} H_i$. Since $h_{\mathcal{A}}$ is constant along W, the equation $\sum_{i \in I} x^i = h_{\mathcal{A}}(W)$ is a linear combination of the equations defining W in $[0,1)^{|I|}$. Using this linear combination on the free term and also evaluated for $x^i = 1$ for every $i \in I$, the result follows.

To end the proof of the theorem, it is sufficient, for any $U \in \mathcal{U}(W)$, to consider the subscheme Z_U of points P that are among the triple points of A_W and for which $\lfloor \sum_{i \in I} e_P^i x^i \rfloor = 2$. □

EXAMPLE 4.7 (The Ceva arrangement $A_1(6)$). Let $\mathcal{A}$ be the Ceva arrangement of degree 6 with three double points and four triple points. Let S be the normalisation of the abelian covering of $\mathbb{P}^2$ branched along $\mathcal{A}$ with $H_\infty \subset \mathcal{A}$ and group $(\mathbb{Z}/n\mathbb{Z})^5$. Then

$$q(S) = \frac{5(n-2)(n-1)}{2}.$$

These surfaces are introduced by F. Hirzebruch in [7]. If $n = 5$, the irregularity was computed by M.-N. Ishida in [9]. The general case was dealt with by A. Libgober in [14].

The sub-arrangements that may have a non-zero contribution in the formula (14) are either the pencil sub-arrangement $\mathcal{A}_P$ of a triple point P, or the arrangement $\mathcal{A}$. Now,

$$h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3 + 2/3 \deg \mathcal{A}_P) \otimes \mathcal{J}(2/3 \cdot \mathcal{A}_P)\big) = h^1(\mathbb{P}^2, \mathcal{I}_P(-1)) = 1$$

and, if Z denotes the support of the triple points,

$$h^1\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3+2/3 \deg \mathcal{A}) \otimes \mathcal{J}(2/3\cdot\mathcal{A})\big) = h^1(\mathbb{P}^2, \mathcal{I}_Z(1)) = 1.$$

So,

$$\begin{aligned} q(S) &= \sum_P |W(\mathcal{A}_P)| \cdot h^1(\mathbb{P}^2, \mathcal{I}_P(-1)) + |W(\mathcal{A})| \cdot h^1(\mathbb{P}^2, \mathcal{I}_Z(1)) \\ &= \sum_P |W(\mathcal{A}_P)| + |W(\mathcal{A})|. \end{aligned}$$

$|W(\mathcal{A}_P)|$ counts in how many ways $2n$ can be written as a sum of three integers that vary in $\{0,1,\ldots,n-1\}$. Let us denote this integer by $\sigma_3(2n)$. It follows that $|W(\mathcal{A}_P)| = \sigma_3(2n) = (n-2)(n-1)/2$. As for $|W(\mathcal{A})|$, it counts the number of solutions of

$$\frac{1}{n}\sum_{j=1}^{6} a^j = 4 \quad \text{and} \quad \frac{1}{n}\sum_{H_j \ni P} a^j = 2 \quad \text{for every } P.$$

This means that $a^1+a^2+a^3 = 2n$ and that $a^i = a^j$ if and only if the lines H_i and H_j intersect in a double point of $\mathcal{A}$. Hence $|W(\mathcal{A})| = \sigma_3(2n)$. The result follows.

4.4 The arrangement dual to the arrangement defined by the inflexion points of a smooth cubic

Another arrangement considered in [7] is the dual to the arrangement defined by the inflexion points of a smooth cubic. Let $\mathcal{A}$ be such an arrangement. It has degree 9 and twelve triple points as only singularities. In particular, each line contains four triple points.

PROPOSITION 4.8. *Let S the normalization of the abelian covering of $\mathbb{P}^2$ branched along $\mathcal{A}$ with $H_\infty \subset \mathcal{A}$ and group $\mathbb{Z}/n\mathbb{Z})^8$. Then $q(S) = 8(n-1)(n-2) - 2\delta^0_{n \bmod 3}$, where δ^j_i denotes the Kronecker symbol.*

Remark. The case $n=5$ is treated in [9] and the general case in [14]. In this latter paper the formula for the irregularity is $8(n-1)(n-2)$, lacking the corrective term in case n is divisible by 3.

Proof. By Theorem 4.6 we have to study faces for which the degree of the corresponding subarrangement $\mathcal{A}_W$ is divisible by 3, *i.e.* equals 3, 6 or 9. In the first case, $\mathcal{A}_W$ is a pencil subarrangement with a single triple point. It will be denoted $\mathcal{A}_P$, with P the triple point. If $\sigma_3(2n)$ is as before the number of ways $2n$ can be written as a sum of three integers from the set $\{0,1,\ldots,n-1\}$, then

$$\sum_P |W(\mathcal{A}_P)| \cdot h^1\big(\mathbb{P}^2, \mathcal{I}_P(-1)\big) = 12\sigma_3(2n).$$

In the second case, $\mathcal{A}_W$ is a Ceva subarrangement and it is easy to see that such a subarrangement cannot exist. As for the last case, there are different faces W such that $\mathcal{A}_W = \mathcal{A}$. Let W be determined by nine points among the twelve triple points—at least nine points are needed so that the corresponding h^1 might be non zero. Since any ten among the twelve points impose independent conditions on cubics as soon as the two remaining points lie on a line of the arrangement, we infer that W is determined by nine points such that there is no line of the arrangement containing any two of the remaining three points. Hence, through each of these three points pass three lines of the arrangement. Now, if Z is the union of the nine points that determine W, then $h^1(\mathbb{P}^2, \mathcal{I}_Z(3)) = 1$. If H_1, H_2 and H_3 are the three lines through one of the triple points not in Z, then summing up the conditions for the points of Z lying on each of these three lines, we obtain that

$$6n = 3a^1 + \sum_{j=4}^{9} a^j = 3a^2 + \sum_{j=4}^{9} a^j = 3a^3 + \sum_{j=4}^{9} a^j.$$

Hence a^j is constant for the lines passing through each of the three missing points. Let $a(W)$, $a'(W)$ and $a''(W)$ be these three constant values. By the preceding equalities,

$$a(W) + a'(W) + a''(W) = 2n. \tag{15}$$

It follows that

$$q(S) = \sum_P |W(\mathcal{A}_P)| \cdot h^1\big(\mathbb{P}^2, \mathcal{I}_P(-1)\big) + \sum_{\substack{W \text{ given} \\ \text{by 9 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{J}(U \cdot \mathcal{A})(3))$$
$$+ \sum_{\substack{W \text{ given} \\ \text{by 10 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{J}(U \cdot \mathcal{A})(3)) + |W(\mathcal{A})| \cdot h^1(\mathbb{P}^2, \mathcal{J}(2/3 \cdot \mathcal{A})(3)),$$

since if a face is defined by eleven points, using (15), it will be defined by all twelve in fact. Moreover, in the two middle sums, $h^1(\mathbb{P}^2, \mathcal{J}(U \cdot \mathcal{A})(3)) = h^1(\mathbb{P}^2, \mathcal{I}_{Z(U)}(3)) = 1$. Indeed, if $U \in \mathcal{U}(W)$ and W is defined by a set Z of nine triple points as above, the subscheme $Z(U)$ is the union of Z and of either one or two more points, depending on the comparison of $3a(W)$, $3a'(W)$ and $3a''(W)$ with $2n$.

In the hereafter lemma it is shown that $h^1(\mathbb{P}^2, \mathcal{J}(2/3 \cdot \mathcal{A})(3)) = 2$. From the preceding considerations and since there are exactly four groups of three points such that there is no line of the arrangement containing any two among the three points,

$$\sum_{\substack{W \text{ given} \\ \text{by 9 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{I}_{Z(U)}(3)) + \sum_{\substack{W \text{ given} \\ \text{by 10 points}}} \sum_{U \in \mathcal{U}(W)} |U| \cdot h^1(\mathbb{P}^2, \mathcal{I}_{Z(U)}(3))$$
$$= \sum_{\substack{W \text{ given} \\ \text{by 9 points}}} \big(|W| - \delta^0_{n \bmod 3} |W(\mathcal{A})|\big) = 4(\sigma_3(2n) - \delta^0_{n \bmod 3}).$$

The corrective term $\delta^0_{n \bmod 3}$ is given by the fact that if n is divisible by 3, then the point in W corresponding to the case $a(W) = a'(W) = a''(W) = 2n/3$ is to be considered in the face $W(\mathcal{A})$. We conclude that

$$q(S) = 12\sigma_3(2n) + 4(\sigma_3(2n) - \delta^0_{n \bmod 3}) + 2\delta^0_{n \bmod 3} = 8(n-1)(n-2) - 2\delta^0_{n \bmod 3}.$$

□

Lemma 4.9. $h^1(\mathbb{P}^2, \mathcal{J}(2/3 \cdot \mathcal{A})) = 2$.

Proof. Let Z denotes the twelve triple points of $\mathcal{A}$. We apply the trace-residual exact sequence to the three lines H_1, H_2 and H_3 that pass through one of the triple points. Let P and P' be the points not lying on these lines. The exact sequences are

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_1} Z}(2) \longrightarrow \mathcal{I}_Z(3) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_2}(\mathrm{Res}_{H_1} Z)}(1) \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_1} Z}(2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{I}_{P+P'} \longrightarrow \mathcal{I}_{\mathrm{Res}_{H_2}(\mathrm{Res}_{H_1} Z)}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \longrightarrow 0,$$

since $\deg \mathrm{Res}_{H_2}(\mathrm{Res}_{H_1} Z) = 3$ and $\mathrm{Res}_{H_3}\big(\mathrm{Res}_{H_2}(\mathrm{Res}_{H_1} Z)\big) = P \cup P'$. It follows that

$$h^1(\mathbb{P}^2, \mathcal{I}_Z(3)) = h^1(\mathbb{P}^2, \mathcal{I}_{\mathrm{Res}_{H_2}(\mathrm{Res}_{H_1} Z)}(1)) = h^1(\mathbb{P}^2, \mathcal{I}_{P\cup P'}) + h^2(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 2.$$

□

4.5 The arrangement associated to the Hesse pencil

Let $\mathcal{A}$ be the line arrangement associated to the Hesse pencil. It is composed by the lines of the four singular fibres of the pencil generated by a smooth elliptic curve and its Hessian. It has degree 12, and twelve double points and nine quadruple points as singularities. The double points correspond to intersection points of lines from the same singular fibre.

Proposition 4.10. *Let S the normalization of the abelian covering of $\mathbb{P}^2$ branched along $\mathcal{A}$ with $H_\infty \subset \mathcal{A}$ and group $(\mathbb{Z}/n\mathbb{Z})^{11}$. Then*

$$q(S) = \frac{(n-1)(61n^2 + 97n - 378)}{6}.$$

Proof. Theorem 3.9 must be used here. The relevant values for each singular point of $\mathcal{A}$ are 2 and 3. The distinguished faces appearing in the formula for the irregularity are of the following types:

1) W_P associated to $(P, 2)$ for each singular point P. The corresponding term in the right hand member of (13) equals $\sigma_4(2n)$ since the conditions are

$$\frac{1}{n}\sum_{j=1}^{12} a^j = 2 \quad \text{and} \quad \frac{1}{n}\sum_{H_j \ni P} a^j = 2.$$

2) W_P associated to $(P, 3)$ for each singular point P. Here the corresponding term equals $\sigma_4(3n)$.

3) $W_{\mathcal{B}}$, with $\mathcal{B}$ a Ceva subarrangement. The face $W_{\mathcal{B}}$ is associated to the singular points of $\mathcal{B}$ seen in $\mathcal{A}$, with relevant value 2 for each one of them. There are 54 such subarrangements, one for each choice of two fibers and two by two components in each fiber—such a choice determines the four triple points of $\mathcal{B}$. As for the terms corresponding to $W_{\mathcal{B}}$ in the formula for the irregularity, let $H_1, \ldots, H_6$ be the lines of $\mathcal{B}$ and let $H_7, \ldots, H_{10}$ be the remaining lines through its triple points. Furthermore we suppose that H_i and H_j

intersect in a double point if and only if $i+j=7$. The defining conditions of $W_{\mathcal{B}}$ are the four equalities corresponding to the triple points:

$$\frac{1}{n}(a^1+a^2+a^3+a^7)=2 \quad \text{and so on, plus} \quad a^{11}=a^{12}=0.$$

The corresponding cohomology group in the formula (7) is non trivial if and only if $h_{\mathcal{A}}(W_{\mathcal{B}}) = \sum_{j=1}^{10} a^j/n = 4$. Summing these four conditions for the four points, we get

$$2\sum_{j=1}^{6} a^j + \sum_{k=7}^{10} a^k = 8n.$$

We conclude that $W_{\mathcal{B}}$ must be defined by $a^7 = \cdots = a^{12} = 0$, $a^1 = a^6$, $a^2 = a^5$ and $a^3 = a^4$, and $a^1 + a^2 + a^3 = 2n$. Hence $|W_{\mathcal{B}}| = \sigma_3(2n)$.

4) W defined by all nine singular points: six with relevant value 2, and the remaining three with relevant value 3. There are three lines H_j that do not pass through the points whose relevant value is 3 and do not intersect in a point. There are 72 such possibilities, $\binom{4}{3}\cdot 3\cdot 3\cdot 2$—$\binom{4}{3}$ choices for the fibres with distinguished components, 3 choices for the distinguished component of the first fibre, 3 for the second and 2 for the third. But, applying the trace-residual exact sequence with respect to the three components, we see that h^1 vanishes.

5) W defined by all nine singular points: six with relevant value 3, and the remaining three with relevant value 2—the configuration obtained from the preceding one by exchanging 2 with 3. As before, $h^1 = 0$ too.

6) W defined by all nine singular points with relevant value 2. Again h^1 does not vanish if and only if $h_{\mathcal{A}}(W) = 6$. Hence the linear system defining W becomes

$$\frac{1}{n}\sum_{j=1}^{12} a^j = 6 \quad \text{and} \quad \frac{1}{n}\sum_{H_j \ni P} a^j = 2 \quad \text{for every singular point } P.$$

Summing up the conditions imposed by the multiple points yields

$$3\sum_{j=1}^{12} a^j = 9\cdot 2n,$$

hence $\sum_{H_j\ni P} a^j = 2n$ for every P. But then

$$6n = \sum_{P\in H_{j_0}}\sum_{H_j\ni P} a^j = 3a^{j_0} + \sum_{\substack{H_j \text{ not a component of the fibre} \\ \text{that contains } H_{j_0}}} a^j.$$

Hence a^j is constant along each special fibre of the Hesse pencil and $|W| = \sigma_4(2n)$.

7) W defined by all nine singular points with relevant value 3. Arguing as in the previous case, $h_{\mathcal{A}}(W) = 9$ and hence we have

$$\frac{1}{n}\sum_{j=1}^{12} a^j = 9 \quad \text{and} \quad \frac{1}{n}\sum_{j\in\alpha} a^j = 3 \quad \text{for every } \alpha,$$

and eventually $|W| = \sigma_4(3n)$. For the computation of the h^1 in this case, the trace residual sequence gives

$$0 \longrightarrow \mathcal{I}_{\sum P}(3) \xrightarrow{u} \mathcal{I}_{\sum 2P}(6) \xrightarrow{r} \mathcal{O}_E \longrightarrow 0,$$

where E is a smooth cubic from the Hesse pencil. Now $h^0(\mathbb{P}^2, \mathcal{I}_{\sum P}(3)) = 2$ and $h^0(\mathbb{P}^2, \mathcal{I}_{\sum 2p_\alpha}(6)) \geq 3$, hence $h^0 r$ is surjective yielding that $h^1(\mathbb{P}^2, \mathcal{I}_{\sum 2p_\alpha}(6)) = 2$.

Summing up,

$$\begin{aligned} q(S) &= 9 \cdot \sigma_4(2n) + 9 \cdot \sigma_4(3n) + 54 \cdot \sigma_3(2n) + \sigma_4(2n) + 2 \cdot \sigma_4(3n) \\ &= 10\, \frac{(n-1)(5n^2 - n - 12)}{6} + 11\, \frac{(n-1)(n-2)(n-3)}{6} + 54\, \frac{(n-1)(n-2)}{2} \\ &= \frac{(n-1)(61n^2 + 97n - 378)}{6}. \end{aligned}$$

□

4.6 General multiple planes

The last example we would like to consider is one that makes use of Theorem 3.9 in its full generality. Let $\mathcal{A}$ be the Ceva's arrangement with the lines $C_1, \ldots, C_6$ such that C_i and C_j determine a double point if and only if $i + j = 7$. Let S' be the $(\mathbb{Z}/5\mathbb{Z})^3$-abelian covering of $\mathbb{P}^2$ defined by the reduced building data

$$\begin{aligned} 5L_{\chi_1} &\sim 3C_2 + C_3 + C_6 \\ 5L_{\chi_2} &\sim 2C_1 + 2C_2 + C_4 \\ 5L_{\chi_3} &\sim C_1 + 3C_3 + C_5. \end{aligned}$$

It is one of the examples considered by M.-N. Ishida in [9, §6], with $q(S) = 10$, where S is the normalization of S'. In [9] it is shown that this surface is a quotient of the Hirzebruch surface constructed as an $(\mathbb{Z}/5\mathbb{Z})^5$-abelian covering of the plane, by the group $(\mathbb{Z}/5\mathbb{Z})^2$. It also verifies $c_1^2 = c_2$. Moreover it is asserted that the surface is isomorphic to the one constructed by M. Inoue (see [8]) from the elliptic modular surface of level 5.

Let us show how the irregularity might be computed using Theorem 3.9. There are non-reduced components in the branch locus and C_∞ is taken to be C_6. We have

$$q(S) = \sum_{P \text{ triple point}} |W(\mathcal{A}_P)|_5^M + |W(\mathcal{A})|_5^M,$$

where

$$M = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and $|W|_5^M = \operatorname{card} \varphi^{-1}(W \cap [0,1)^6)$, with $\varphi : [0,1)^3 \cap (1/5\mathbb{Z})^3 \to [0,1)^6$ defined by

$$\varphi\left(\frac{a^1}{5}, \frac{a^2}{5}, \frac{a^3}{5}\right) = \left(\left\langle \sum_{j=1}^{3} m_j^1 \frac{a^j}{5} \right\rangle, \ldots, \left\langle \sum_{j=1}^{3} m_j^6 \frac{a^j}{5} \right\rangle\right).$$

Here as before, $\mathcal{A}_P$ is the pencil subarrangement determined by the triple point P. An easy computation gives $|W(\mathcal{A}_P)|_5^M = 2$ for every triple point. Furthermore, since the equations

$$\left\langle \frac{2a^2+a^3}{5} \right\rangle = \left\langle \frac{a^1}{5} \right\rangle, \quad \left\langle \frac{3a^1+2a^2}{5} \right\rangle = \left\langle \frac{a^3}{5} \right\rangle, \quad \left\langle \frac{a^1+3a^3}{5} \right\rangle = \left\langle \frac{a^2}{5} \right\rangle$$

and $\langle a^1/5\rangle + \langle a^2/5\rangle + \langle a^3/5\rangle = 2$ lead to the only solutions $(2,4,4)$ and $(4,3,3)$, it follows that $|W(\mathcal{A})|_5^M = 2$ also. Hence the irregularity is 10.

A Technical result

In the proof of Theorem 3.9 we used two technical results that enabled us to describe the reduced building data of the normalization of a standard covering—which is also a standard covering (see [19, Corollary 3.1])—in terms of the initial reduced building data. The first result was Proposition 3.2 in [17]. The second is somehow similar and deals with the fourth step in the normalization algorithm presented in [19]. It is the step peculiar to the abelian situation.

Proposition A.1. *Let X be smooth and let $\pi\colon Y \to X$ be a standard abelian covering determined by the set of reduced building data $\mathcal{L}_{\chi_j}$ and B_f, $1 \le j \le s$ and $f \in \mathfrak{F}$. Let C be a multiplicity 1 component of both B_f and B_g, i.e. $B_f = C + R_f$ and $B_g = C + R_g$. After the normalization procedure has been applied to C and $Y' \to Y$ is the new surface, if $\chi = \chi_1^{a_1}\cdots\chi_s^{a_s}$, then*

$$\begin{aligned} L'_\chi \sim \sum_{j=1}^s a_j L_{\chi_j} &- \left\lfloor \sum_{j=1}^s a_j \left(\frac{f(\chi_j)^\bullet}{m_f} + \frac{g(\chi_j)^\bullet}{m_g} \right) \right\rfloor C \\ &- \left\lfloor \sum_{j=1}^s \frac{a_j f(\chi_j)^\bullet}{m_f} \right\rfloor R_f - \left\lfloor \sum_{j=1}^s \frac{a_j g(\chi_j)^\bullet}{m_g} \right\rfloor R_g - \sum_{h \ne f,g} \left\lfloor \sum_{j=1}^s \frac{a_j h(\chi_j)^\bullet}{m_h} \right\rfloor B_h. \end{aligned}$$

Proof. Assume that $f : \widehat{G} \to \mathbb{Z}/m_f$ and that $g : \widehat{G} \to \mathbb{Z}/m_g$. Let d and m be the greatest common divisor of, and respectively the smallest common multiple of m_f and m_g. If $\varphi : \mathbb{Z}/m_f \times \mathbb{Z}/m_g \to \mathbb{Z}/m$ is defined by $\varphi(1,0) = m_g/d$ and $\varphi(0,1) = m_f/d$, then set $f' : \widehat{G} \to \mathbb{Z}/m_{f'}$ the morphism defined by the composition

$$\widehat{G} \xrightarrow{f\times g} \mathbb{Z}/m_f \times \mathbb{Z}/m_g \xrightarrow{\varphi} \operatorname{Im}\overline{\varphi} \xrightarrow{\iota} \mathbb{Z}/m_{f'}$$

where $\overline{\varphi}$ is the morphism $\varphi \circ (f \times g)$ and ι the isomorphism defined by $\iota(m/m_{f'}) = 1$. The normalization of Y along C is constructed by modifying the covering data as follows:

$$L'_\chi \sim \begin{cases} L_\chi - C, & \text{if } \dfrac{f(\chi)^\bullet}{m_f} + \dfrac{g(\chi)^\bullet}{m_g} \ge 1 \\ L_\chi, & \text{otherwise} \end{cases}$$

and

$$B'_f \sim B_f - C, \quad B'_g \sim B_g - C, \quad B'_{f'} \sim B_{f'} + C, \quad B'_h \sim B_h \quad \text{for} \quad h \neq f, g, f'.$$

Applying these modifications to (6) gives

$$\begin{aligned} L'_\chi &\sim \sum_j a_j L'_{\chi_j} - \sum_{h \in \mathfrak{F}} \left\lfloor \sum_j \frac{a_j h(\chi_j)^\bullet}{m_h} \right\rfloor B_h \\ &\sim {\sum_i}' a_i (L_{\chi_i} - C) + {\sum_k}'' a_k L_{\chi_k} - \left\lfloor \sum_j \frac{a_j f(\chi_j)^\bullet}{m_f} \right\rfloor R_f - \left\lfloor \sum_j \frac{a_j g(\chi_j)^\bullet}{m_g} \right\rfloor R_g \\ &\quad - \left\lfloor \sum_j \frac{a_j f'(\chi_j)^\bullet}{m_{f'}} \right\rfloor (B_{f'} + C) - \sum_{h \neq f, g, f'} \left\lfloor \sum_j \frac{a_j h(\chi_j)^\bullet}{m_h} \right\rfloor B_h, \end{aligned}$$

where the sum $\sum'$ runs over those i's for which $f(\chi)^\bullet / m_f + g(\chi)^\bullet / m_g \geq 1$ and $\sum''$ over the other k's. To prove the result it is sufficient to show that

$${\sum_i}' a_i + \left\lfloor \sum_j \frac{a_j f'(\chi_j)^\bullet}{m_{f'}} \right\rfloor = \left\lfloor \sum_{j=1}^s a_j \left(\frac{f(\chi_j)^\bullet}{m_f} + \frac{g(\chi_j)^\bullet}{m_g} \right) \right\rfloor .$$

But

$$\begin{aligned} \left\lfloor \sum_j \frac{a_j f'(\chi_j)^\bullet}{m_{f'}} \right\rfloor &= \left\lfloor \sum_j \frac{a_j}{m_{f'}} \left(\frac{m_g}{d} f(\chi_j)^\bullet + \frac{m_f}{d} g(\chi_j)^\bullet \right)^\bullet \frac{m_{f'}}{m} \right\rfloor \\ &= \left\lfloor {\sum_i}' + {\sum_k}'' \right\rfloor . \end{aligned}$$

Since for each i in the first sum

$$\left(\frac{m_g}{d} f(\chi_i)^\bullet + \frac{m_f}{d} g(\chi_i)^\bullet \right)^\bullet = \frac{m_g}{d} f(\chi_i)^\bullet + \frac{m_f}{d} g(\chi_i)^\bullet - m$$

and for each k in the second

$$\left(\frac{m_g}{d} f(\chi_k)^\bullet + \frac{m_f}{d} g(\chi_k)^\bullet \right)^\bullet = \frac{m_g}{d} f(\chi_k)^\bullet + \frac{m_f}{d} g(\chi_k)^\bullet,$$

the identity follows. □

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